11. B. C. Sakiadis and I. Coates, "The organic liquids thermal conductivity investigations," A. I. Chem. J., 1, No. 3, 275-283 (1955).
12. T. F. Klimova, "Study of thermophysical properties of complex ether-propionates over a wide range of state parameters," Author's Abstract of Candidate's Dissertation, Groznyi (1978).

QUESTION OF CONSTRUCTING TIKHONOV REGULARIZING ALGORITHMS FOR NON-ONE-DIMENSIONAL INVERSE PROBLEMS OF HEAT CONDUCTION

V. B. Glasko and E. E. Kondorskaya

UDC 517.9:536.2

Tikhonov regularizing algorithms are formulated and given a foundation for the multidimensional problem of determining the density of a finite heat source. Uniqueness of the solution is studied for one of the formulations of the problem.

1. One of the inverse problems in the theory of heat conduction is that of determining the characteristics of a finite heat source by means of certain data about the temperature field it generates — in particular, the problem of detecting defects in the lining of exothermal production reactors that originate during their utilization.

Since the inverse problem being considered is among the incorrect ones, a Tikhonov regularizing algorithm (RA) must be utilized for its solution [1]. The concept of an RA includes a broad class of stable algorithms and some have been developed or realized in [2, 3] for heat conduction problems. The most extensive domain of application, including the topic under consideration, has a general A. N. Tikhonov regularizing algorithm based on the solution of a certain auxiliary variational problem for a "smoothing" functional. The stabilizer [1] is the element of such a functional that governs the stability.

This algorithm was realized in [4, 5] for the solution of different one-dimensional inverse problems. The problem of determining a heat source is multidimensional. In this case the stabilizer construction given a mathematical foundation is sufficiently complicated, and the question occurs of a possible simplification so well founded as to denote a saving in machine computations in practice.

In this paper we limit ourselves mainly to a well-founded formulation of simplifications associated with the finiteness of the desired function. Here the starting point is the fact that according to [6] the problem of minimizing the Tikhonov smoothing functional governing the general RA can also include quantitative information about the desired solution of the problem. The boundary conditions for the desired function can be such information for the problem of a finite source.

Let us note that the general Tikhonov RA and the modifications proposed below are considered for utilization with an electronic computer, which corresponds to modern possibilities of engineering practice and its prospects.

2. Let us first consider the problem of a concentrated heat source, in which example we clarify the question of the correctness of the formulation as well as the possibility of simplifying the stabilizer. Here and henceforth, we limit ourselves to a consideration of an infinitely extended spatial model, which does not limit the generality of the algorithms being formulated for problems of appropriate dimensionality.

Let the temperature field $u(x, t)$, $x = (x_1, x_2) \in E_2$ satisfy the conditions

$$k\Delta u + F(x,\ t) = c\rho \frac{\partial u}{\partial t},\ |x| < +\infty,\ 0 < t \leqslant T,\ |u| \underset{|x|\to\infty}{\to} 0,\ u|_{t=0} = 0, \tag{1}$$

where $F(x, t) = f(t)\delta(x, x_0)$, $\delta(x, x_0)$ is the Dirac function, and x_0 is the source location $f(t) \equiv 0$ for $t \leqslant 0$.

M. V. Lomonosov Moscow State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 43, No. 4, pp. 631-637, October, 1982. Original article submitted July 7, 1981.

0022-0841/82/4304-1139$07.50

The desired quantities in the inverse problem in which we are interested are x_0 and $f(t)$ that are related to the exact temperature field by the equation

$$u(x,\ t) = \int_0^t G_2(|x-x_0|^2,\ t-\tau) f(\tau)\, d\tau \equiv A(x,\ t,\ x_0,\ f) \equiv A(x_0,\ f), \tag{2}$$

where G_2 is the Green's function for a plane, $A(x, t, x_0, f)$ and $A(x_0, f)$ are the abbreviated notation for the nonlinear integral operator on the right.

If x_0 is given, then for any fixed $x \neq x_0$ Eq. (2) is an integral equation of convolution type and under the assumption of the continuity of $f(t)$ has a unique solution for $f(t)$ [7]. On the other hand, if $f(t)$ is given and $x_0 = (x_{10}, x_{20})$ is desired, then for fixed $x_s \neq x_0$, $s = 1, 2$ and $t = t_1 > 0$, (2) is a system of transcendental equations in x_{10}, x_{20}. Setting $|x_s - x_0|^2 \equiv \xi$ (in the expression G_2), we note that

$$\varphi(\xi) = \int_0^{t_1} G_2(\xi,\ t_1-\tau) f(\tau)\, d\tau$$

is a continuous, monotonically decreasing function ($\varphi(\xi) \to 0$ as $\xi \to \infty$ and ($\varphi(\xi) \to \infty$ as $\xi \to 0$), and hence, the equation $\varphi(\xi) = u(x_s, t_1)$ has a unique solution ξ_s for every s.

However, a pair of values ξ_1 and ξ_2 determines a pair of points of a plane for an a priori existence of the solution (the intersection of pairs of circles); for a single-valued selection of one of them it is sufficient to give the value of the temperature at still another point not coincident with the first two.

Without going into the question of the single-valued determination of the full set of x_0, $f(t)$, let us note that the analysis performed permits the idea that the minimal information by which the single-valued solution of the problem can be computed is the assignment of the temperature at the time $t_1 > 0$ at any three points not on one line (and not agreeing a priori with x_0) as well as the temperature at one of these points as a function of time. We take this observation as a working hypothesis for the formulation of the algorithm.* If the solution of the problem is unique for such data, then utilization of the RA will afford a possibility of obtaining an arbitrarily exact approximation to the appropriate solution of the problem for a sufficiently small error in the input data.

If the temperature is given inaccurately (measured)

$$\rho^2(\bar{u},\ \tilde{u}) \equiv \int_0^T (\bar{u}(x_1,\ t) - \tilde{u}(x_1,\ t))^2 dt + \sum_{s=2}^{3} p(\bar{u}(x_s,\ t_1) - \tilde{u}(x_s,\ t_1))^2 \leqslant \delta^2$$

(where p is a dimensionality factor and δ is a measure of the error), then (2) has no solution because of the continuity of G_2 in x and t and the simultaneously stochastic discontinuity of the given function; in other words, the first condition for correctness of the problem formulation is spoiled and (2) is only a "conditional" equation. But even in the presence of a solution (for a certain $\tilde{u}$) its instability relative to the variation $\tilde{u}$ is evident ($\tilde{u}(x, t)$ is the integral effect of the function $f(t)$ which is not responsive to its abrupt local changes). In other words the third condition for correctness in the formulation of the problem is spoiled.

We use qualitative information about the desired solution for the possible correct formulation of the problem. Let it be known that $f(t)$ is a smooth function bounded in the interval of observations $(0, T)$. These conditions are automatically assured (in the general RA) by the introduction of the "stabilizer" functional [1]:

$$\Omega[f] \doteq \int_0^T (f'^2(t) + f^2(t))\, dt,$$

where no quantitative information about $f(t)$ (in particular, the constants bounding Ω) is required. The evident information about the boundedness of the vector x_0 also exactly generates the stabilizer:

*Let us note that the general RA assures convergence to a certain completely determined solution even in the case when the solution is not unique [1].

$$\Omega[x_0] \equiv q \sum_{i=1}^{2} x_i^2$$

(q is a dimensionality factor), which within the framework of the general RA assures the boundedness of x_0 without additional quantitative information.

Now, the Tikhonov regularizing algorithm can be formulated as any algorithm to minimize the smoothing functional which is in our case

$$F_\alpha(x_0, f) \equiv \rho^2(Af, \tilde{u}) + \alpha(\Omega[f] + \Omega[x_0]), \tag{3}$$

where

$$\rho^2(Af, \tilde{u}) \equiv \int_0^T (A(x_1, t, x_0, f) - \tilde{u}(x_1, t))^2 dt + p \sum_{s=2}^{3} (A(x_s, t_1, x_0, f) - \tilde{u}(x_s, t))^2.$$

The value $\alpha = \alpha(\delta)$ can be selected during the computation from the residual condition [1, 8]: $\rho^2(A(x_0^\alpha, f^\alpha), \tilde{u}) = \delta^2$ if (x_0^α, f^α) minimizes (3) for a certain α.

The finite-difference approximation of f, f' and the elements of (3) result in a variational problem for a function of many variables $F_\alpha(x_{10}, x_{20}, f_1, \ldots, f_m)$ but we do not consider here the detailed description of the procedure for successive minimization (by a reduction of α) described in [9] for a similar type of problem where the effectiveness of the algorithm is also shown.

Let us turn to the question of simplifying the stabilizer for f(t). We note that the effectiveness of the general RA is assured by the condition (γ): the set of functions f(t) for which $\Omega[f] \leqslant d$ (for some $d > 0$) is compact in C[0, T]. This latter means (without limiting the generality) that any infinite sequence $f_n(t)$ from this set will converge uniformly in [0, T] to a certain function $\hat{f}(t)$. In particular, for sufficiently small δ, $\max|f^{\alpha(\delta)}(t) - \bar{f}(t)|$ is arbitrarily small if $(\bar{x}_0, \bar{f}(t))$ is the exact solution of the formulated problem. For the stabilizer selected the condition (γ) is satisfied automatically for the minimization (3). If we set

$$\Omega[f] = \Omega_0[f] = \int_0^T f'^2(t)\, dt$$

(without using any information about f(t) except smoothness), then the condition (γ) is not satisfied so that the convergence of the approximation to the exact solution is not assured.

However, in the problem under consideration the condition f(0) = 0 is natural additional information about f(t). As is shown in [6], the set of functions f(t) for which $\Omega_0[f] \leqslant d$ and f(0) = 0 simultaneously, is compact in C[0, T]. Hence, the algorithm to minimize the functional $\rho^2(Af, \tilde{u}) + \alpha(\Omega_0[f] + \Omega[x_0])$ in a set with the constraint f(0) = 0 (realizable without difficulty in a finite approximation) remains Tikhonov regularizing for the problem under consideration.

3. The simplification of the stabilizer executed in Sec. 2 can turn out to be especially effective for multidimensional problems.

Let us consider the problem of a finite heat source with the density f(x, t), $x = (x_1, \ldots, x_n) \in E_n$, n = 1, 2, 3, where $f(x, t) \equiv 0$ outside the domain $g_n \subset E_n$ and for $t \leqslant 0$.

Taking account of the results of investigating the single-valuedness of the mapping $u(x, t) \to f(x, t)$ in the preceding problem and without examining the corresponding problem for a continuous density, let us consider assignment of the temperature u(x, t) in a certain domain without an intersection (in the space variables) with $g_n (t \in [0, T])$* to be reasonable minimal information for the determination of f(x, t).

Then a problem analogous to (1) results in the following (conditional for inaccurately given $\tilde{u}(x, t)$) integral equation:

*Assignment of u(x, t) everywhere in E_n for $t > 0$ evidently determines f(x, t) single-valuedly, but we are interested in minimal information.

$$\tilde{u}(x, t) = \int_{g_n}\int_0^t G_n(|x-\xi|^2, t-\tau) f(\xi, \tau)\, d\xi d\tau \equiv Af, \tag{4}$$

where G_n is the Green's function for a space of appropriate dimensionality, and Af is, as before, the abbreviated notation for the integral operator.

Let

$$\rho^2(\bar{u}, \tilde{u}) \equiv \int_{D_n}\int_0^T (\bar{u}(x, t) - \tilde{u}(x, t))^2\, dxdt \leqslant \delta^2.$$

Then the general Tikhonov RA can be formulated for a previous matching of α and δ, in the form

$$\inf F_\alpha[f],\ F_\alpha[f] \equiv \rho^2(Af, \tilde{u}) + \alpha\Omega[f], \tag{5}$$

where $\Omega[f]$ is the stabilizer.

The stabilizer for the multidimensional problem under consideration (the functional satisfying the condition (γ) in Sec. 2) has the form

$$\Omega[f] = \int_{g_n}\int_0^T \left(\sum_{k=0}^{p} \sum_{k_1+\dots+k_{n+1}=k} \left(\frac{\partial^k f}{\partial x_1^{k_1}\dots\partial x_n^{k_n}\partial t^{k_{n+1}}}\right)^2\right) dxdt \tag{6}$$

under the condition that $2p > n + 1$ [10]; $p \geqslant 2$ for a line ($n = 1$) and a plane ($n = 2$), respectively, while $p \geqslant 3$ for three-dimensional space ($n = 3$). Here the presence of all derivatives of the same order (including mixed) in the sum is essential.

Let us show that under definite additional information about the source, the simplified functional

$$\Omega_0[f] = \int_{g_n}\int_0^T \left(\frac{\partial^{n+1} f}{\partial x_1\dots\partial x_n\partial t}\right)^2 dxdt. \tag{7}$$

can be used as stabilizer in the multidimensional problem under consideration. Let g_n be a generalized parallelepiped, $a_i \leqslant x_i \leq b_i$, $i = 1, 2, \dots, n$, and let the exact solution of the problem satisfy the conditions (β): $f(a_i, t) = 0$ (or $f(b_i, t) = 0$), $i = 1, 2, \dots, n$; $\bar{f}(x, 0) = 0$ is sufficiently natural for a finite source. Let K denote the set of functions $f(x, t)$ satisfying the conditions (β) and possessing continuous derivatives corresponding to (7).

<u>THEOREM.</u> For any $d > 0$ the set $\{f(x, t)\}$ satisfying the conditions $\Omega_0[f] \leq d$, $f \in K$ is compact in $C(g_n \times [0, T])$.

Let us limit ourselves to the proof of this assertion for $n = 1$ and the case $f(a_i, t) = 0$ (for $n > 1$ and the case $f(b_i, t) = 0$ the proof is analogous).

We see the equipotential continuity of the functions satisfying the conditions of the theorem. For any pair of points (x_1, t_1) and (x_2, t_2) from the domain under consideration, $|f(x_1, t_1) - f(x_2, t_2)| \leqslant |f(x_1, t_1) - f(x_1, t_2)| + |f(x_1, t_2) - f(x_2, t_2)| \equiv \Delta_1 + \Delta_2$. But

$$\Delta_1 = |f(x_1, t_1) - f(x_1, t_2)| \leqslant \left|\int_{t_1}^{t_2} \frac{\partial f}{dt}(x_1, t)\, dt\right| \equiv \left|\int_{t_1}^{t_2}\left(\frac{\partial f}{\partial t}(x_1, t) - \frac{\partial f}{\partial t}(a_1, t)\right) dt\right|,$$

since $f(a_1, t) = 0$, which means $\partial f/\partial t(a_1, t) = 0$. Hence

$$\Delta_1 \leqslant \left|\int_{t_1}^{t_2}\int_{a_1}^{x_1} \frac{\partial^2 f}{\partial x\partial t}\, dxdt\right| \leqslant \left(\left|\int_{t_1}^{t_2}\int_{a_1}^{x_1}\left(\frac{\partial^2 f}{\partial x\partial t}\right)^2 dxdt\right| \cdot |t_1 - t_2| \cdot |x_1 - a_1|\right)^{1/2} \leqslant (d(b_1 - a_1))^{1/2}\sqrt{|t_1 - t_2|} \equiv d_1\sqrt{|t_1 - t_2|}.^*$$

Analogously, because of the condition $f(x, 0) = 0$, which means $\partial f/\partial x(x, 0) = 0$ $\Delta_2 \leqslant d_2\sqrt{|x_1 - x_2|}$, where $d_2 \equiv (dT)^{1/2}$. Using the notation $d_3 = \max(d_1, d_2)$ we have $|f(x_1, t_1) - f(x_2, t_2)| \leqslant d_3(\sqrt[4]{(t_1 - t_2)^2} + \sqrt[4]{(x_1 - x_2)^2}) \leq 2d_3\sqrt[4]{(t_1 - t_2)^2 + (x_1 - x_2)^2}$, from which indeed follows the equipotential continuity of the functions in the conditions of the theorem.

We see their uniform boundedness. For any point $(x, t) \in g_1 \times [0, T]$ $|f(x, t)| = |f(x, t) - f(x, 0)| \leqslant 2d_3\sqrt{T}$ according to the preceding estimate, and the uniform boundedness is proved.

*Here the Cauchy–Bunyakovskii inequality is used.

According to the Artsel theorem [11], compactness of the set in the sense of uniform convergence follows from the equipotential continuity and uniform boundedness of the functions. The theorem is proved.

Therefore, for the problem under consideration the regularized approximation $f^{\alpha(\delta)}(x, t)$ to $\bar{f}(x, t)$ can be found from the following variational problem

$$\inf F_\alpha[f],\ F_\alpha[f] = \rho^2(Af,\ \tilde{u}) + \alpha\Omega_0[f],\ f \in K, \tag{8}$$

where $\alpha(\delta)$ is determined by the condition $\rho^2(Af^\alpha, \tilde{u}) = \delta^2$.

Let us note that for practical purposes the Euler equation following from (8) can be used, where it is taken into account in a natural manner that the approximation belongs to the set K. Here, however, more detailed information about the desired solution must be used as compared with the method of direct minimization and the theorem. The Euler equation is obtained by the ordinary variational method [1], and in particular, has the following form for n = 1

$$\int_{g_1}\int_0^T K(\eta,\ \xi,\ t,\ \tau) f(\xi,\ \tau)\, d\xi d\tau + \alpha \frac{\partial^4 f}{\partial\eta^2 \partial t^2}(\eta,\ t) = b(\eta,\ t) \tag{9}$$

under the conditions $f(a, t) = f(b, t) = 0$, $f(x, 0) = 0$ which are natural for problems with a finite source, and the additional condition $f'(x, T) = 0$ which can turn out to be approximate. Here

$$K(\eta,\ \xi,\ t,\ \tau) \equiv \int_{D_1}\int_0^T G_1(x,\ \eta,\ \theta,\ t)\, G_1(x,\ \xi,\ \theta,\ t)\, dx d\theta,$$

$$b(\eta,\ t) \equiv \int_{D_1}\int_0^T G_1(x,\ \eta,\ \theta,\ t)\, \tilde{u}(x,\ \theta)\, dx d\theta.$$

Finite-difference approximations reduce (9) to a system of linear algebraic equations solved by standard programs.

The theorem established above is true even for spaces of arbitrary dimension ($n > 3$), which has, however, no relationship to the problem under consideration. The simplified functional introduced here (and in Sec. 2) is naturally called a conditional stabilizer.

4. For certain two-dimensional inverse problems a functional that is a formal generalization of $\Omega[f]$ (Sec.2) is sometimes utilized. Namely, if $f = f(x, y, z)$ is a function of three variables defined in the domain g, then

$$\Omega_1[f] \equiv \|f\|^2_{W_2^1} = \int_g \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 + f^2 \right) d\sigma \equiv \int_g ((\Delta f)^2 + f^2)\, d\sigma. \tag{10}$$

It is easy to see that such a functional does not satisfy the condition (γ), and therefore, is not a stabilizer assuring any accuracy of the approximation (for sufficiently small δ) without involving additional information about the desired solution. Indeed, let $(x_0, y_0, z_0) \in g$ and $f = f_\lambda(x, y, z) = r^{-\lambda}$, $r = ((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2)^{1/2}$. Then

$$\Delta f_\lambda = -\frac{\lambda}{r^{\lambda+1}} \left\{ \frac{x - x_0}{r},\ \frac{y - y_0}{r},\ \frac{z - z_0}{r} \right\},$$

and $\Omega_1[f_\lambda]$ is bounded for any λ: $0 < \lambda < 1/2$. However, $f_\lambda(x, y, z) \to \infty$ as $(x, y, z) \to (x_0, y_0, z_0)$ is discontinuous and not bounded in g. Therefore, the set $\{f(x, y, z)\}$ for which $\Omega_1[f] \leqslant d$ cannot be a set of either a uniformly bounded or equipotentially continuous functions. Involvement of additional quantitative information of the boundary-condition type evidently does not alter the situation (an example from the class of "fundamental" functions can be cited [12]).

Utilization of this functional as stabilizer [1, 13] can be given a foundation only by taking more complete account of the specific properties of the solution in the algorithm being developed. Thus, the desired solution of the inverse problem in [13] satisfies the Laplace equation, and this is used together with quantitative constraints of the boundary-conditions type. In such a situation (10) is also a conditional stabilizer.

LITERATURE CITED

1. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solving Incorrect Problems [in Russian], Nauka, Moscow (1979).
2. O. M. Alifanov, "Regularization schemes to solve inverse problems of heat conduction," Inzh.-Fiz. Zh., 24, No. 2, 324-333 (1973).
3. A. N. Tikhonov, N. I. Kulik, I. N. Shklyarov, and V. B. Glasko, "On the results of mathematical modeling of a heat conduction process," Inzh.-Fiz. Zh., 39, No. 1, 5-10 (1980).
4. A. N. Tikhonov and V. B. Glasko, "On the question of methods of determining the temperature of a body surface," Vychisl. Mat. Mat. Fiz., 7, No. 4, 910-914 (1967).
5. V. B. Glasko, N. V. Zakharov, and A. Ya. Kolp, "On restoration of the thermal flux to a body surface for a nonlinear heat conduction process on the basis of the regularization method," Inzh.-Fiz. Zh., 29, No. 1, 60-62 (1975).
6. V. B. Glasko, G. V. Gushchin, and E. D. Mudretsova, "On utilization of drilling data for restoration of the contact shape by using the regularization method," Vychisl. Mat. Mat. Fiz., 14, No. 5, 1272-1280 (1974).
7. V. A. Ditkin and A. P. Prudnikov, Operational Calculus [in Russian], Vysshaya Shkola, Moscow (1975).
8. V. A. Morozov, "On the residual principle for solving operator equations by the regularization method," Zh. Vyshisl. Mat. Mat. Fiz., 8, No. 2, 295-305 (1968).
9. A. N. Tikhonov, V. B. Glasko, and N. I. Kulik, "Regularizing algorithms for nonlinear problems and inverse problem of magnetotelluring sounding," Computation Methods and Programming [in Russian], No. 20, Moscow Univ. Press (1973), pp. 158-174.
10. S. M. Nikol'skii, Approximation of Functions of Many Variables and Imbedding Theorems [in Russian], Nauka, Moscow (1969).
11. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Graylock (1961).
12. V. S. Vladimirov, Generalized Functions [in Russian], Nauka, Moscow (1976).
13. A. N. Tikhonov, V. B. Glasko, and O. K. Litvinenko, "On continuation of a potential towards perturbing masses on the basis of the regularization method," Fiz. Zemli, No. 12, 30-48 (1968).

ANALYSIS OF TEMPERATURE FIELDS OF BODIES IN THE SHAPE OF SHELLS

N. M. Belyaev, V. I. Zavelion,
and A. A. Ryadno

UDC 536.24:539.3

The temperature fields in axisymmetric thick-walled shells with different middle surface shape are investigated.

In performing thermal engineering analyses of metallurgical or power equipment there often occurs the need to solve heat conduction problems for bodies in the shape of shells. In these cases it is natural to use shell theory methods [1-3]. The nonclassical theory of shells [2], whose equations are valid for nonthin shells with rapidly varying geometry and thickness, is used below to determine the nonstationary temperature field of a unified slag car cup.

A slag car cup is an axisymmetric thick-walled shell formed by a spherical segment connected to a hollow truncated cone of linearly varying thickness; hence, the boundary conditions will also be axisymmetric, which permits solution of the problem for the domain displayed in Fig. 1. The slag pouring periods and further heating of the cup up to the time of emptying are considered; it is assumed in the computations that the thermophysical constants of the cup material are independent of the temperature.

300th Anniversary of the Union of Ukraine and Russia Dnepropetrovsk State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 43, No. 4, pp. 637-641, October, 1982. Original article submitted August 14, 1981.

 0022-0841/82/4304-1144$07.50